MTH 203 - Quiz 1 Solutions

1. Does the set $\mathrm{Sym}(n, \mathbb{R})$ of real symmetric $n \times n$ matrices defined by

$$\mathrm{Sym}(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) : A = A^\intercal\}$$

form a group under matrix addition? Explain why or why not.

Solution. We know from class that $M_n(\mathbb{R})$ is a group under matrix addition. Consider a pair of arbitrary symmetric matrices $A, B \in \mathrm{Sym}(n, \mathbb{R})$. Then

$$(A - B)^\intercal = A^\intercal - B^\intercal = A - B,$$

which implies that $A - B \in \mathrm{Sym}(n, \mathbb{R})$. From the Subgroup Criterion (see Lesson Plan 1.1 (vii)), it follows that $\mathrm{Sym}(n, \mathbb{R}) < M_n(\mathbb{R})$. This shows that $\mathrm{Sym}(n, \mathbb{R})$ is also a group under matrix addition.

2. Consider $[k] \in \mathbb{Z}_n$ such that $\gcd(k, n) = 1$. Show that $\mathbb{Z}_n = \langle [k] \rangle$.

Solution. For convenience, we will identify $\mathbb{Z}_n$ with the group

$$C_n = \langle g \rangle = \{1, g, g^2, \ldots, g^{n-1}\}.$$

Under this identification $[k] \in \mathbb{Z}_n$ is identified with the element $g^k \in C_n$. We know from class $o(g^k) = n/\gcd(k, n)$. Suppose that $\gcd(k, n) = 1$. Then we have that $o(g^k) = n$. Hence, it follows that $\langle g^k \rangle = C_n$, and consequently, $\mathbb{Z}_n = \langle [k] \rangle$.

Conversely, suppose that $\langle [k] \rangle = \mathbb{Z}_n$. Then $o([k]) = o(g^k) = n$, which implies that $n/\gcd(k, n) = n$. Hence, $\gcd(k, n) = 1$.